Implementing the Commitment Solution via Discretionary Policy-Making

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Abstract

In a large class of linear-quadratic models with rational expectations, the commitment solution can be implemented by a policy-maker who acts purely under discretion. To show this, we construct discretionary equilibria with payoff-irrelevant state variables. These additional state variables are constructed in a way such that the policy-maker finds it impossible to renege on past promises. On the equilibrium path, the state variables are identical to the Lagrange multipliers associated with the constraints for the non-predetermined variables in the corresponding commitment problem.

Keywords: Discretion, Commitment, LQ RE models.

JEL: E58, E61, C61.

1 Introduction

This paper shows that, for the large class of linear models with rational expectations considered by Blanchard and Kahn (1980) and a quadratic objective function of the policy-maker, the commitment solution can also be implemented by a policy-maker under discretion, i.e. a policy-maker that cannot commit to a specific future behavior but chooses its instrument optimally every period, taking its own future behavior as given.[1,2] Hence the social losses stemming from the time-inconsistency problem in models where current economic variables depend on expectations about future variables can be avoided.[3]

The key to this result is to consider discretionary equilibria that are not Markov-perfect but where all agents including the policy-maker respond to additional state variables that, on the equilibrium path, are identical to the Lagrange multipliers associated with the constraints for the non-predetermined (or forward-looking) variables in the corresponding problem under commitment.[4] As is well-known, when Markov-perfection is imposed, a discretionary policy-maker typically cannot make credible promises to set non-predetermined variables in the future to the values implied by the commitment solution because the policy-maker will later find it optimal to deviate. In the discretionary equilibria constructed in this paper, the policy-maker can credibly promise to set these variables in line with the commitment solution because, in every period, the policy-maker is unable to influence current non-predetermined variables. This is achieved by introducing the additional state variables in a way such that the direct effect of a change in the instrument on the non-predetermined variables is exactly offset by a change in the expectations about future non-predetermined variables. Effectively, this procedure turns non-predetermined variables into predetermined (or backward-looking) ones.

Potential gains from commitment have been identified in the classic literature on the inflation bias (Kydland and Prescott, 1977; Barro and Gordon, 1983). Kydland and Prescott (1977) and Stokey (1989) demonstrate the time-inconsistency of optimal government policies in other fields like taxation or patent protection. In the standard new

[1] The discretionary solution and the commitment solution have been analyzed in Oudiz and Sachs (1985), Backus and Driffill (1986), Currie and Levine (1993), and Söderlind (1999).

[2] Waki et al. (2018) consider the optimal degree of discretion in a framework where society imposes dynamic constraints on the policy-maker.

[3] How the commitment solution of such models can be obtained via a recursive saddle-point functional equation is shown by Marcet and Marimon (2019).

[4] Responding to payoff-irrelevant variables can be optimal in linear-quadratic models of discretionary policy-making if current economic variables depend on expectations about future economic variables and if the payoff-irrelevant variables are defined recursively (Hahn, 2019).

Keynesian model, another time-inconsistency problem arises, the so-called stabilization bias (Clarida et al., 1999; Woodford, 1999). Papers that aim to quantify the gains from commitment for central banks find that they are potentially sizable (Dennis and Söderström, 2006; Levine et al., 2008). As a consequence, it is important to answer the question how the gains from commitment can be achieved.

A recent paper by Debortoli et al. (2018) assumes that the central bank can implement the commitment solution for a given loss function that captures a specific mandate. Debortoli et al. (2014) and Debortoli and Lakdawala (2016) find empirical evidence that the Federal Reserve operates with a high degree of commitment. These papers leave open the question how central banks can implement the commitment solution. The present paper lays out such a mechanism.

A related paper by Hahn (2019) shows the existence of non-Markov-perfect discretionary equilibria and points out the potential for them to be welfare enhancing. In contrast with the present paper, Hahn (2019) focuses exclusively on the canonical new Keynesian model. Moreover, for the class of equilibria considered by Hahn (2019), a discretionary central bank can never implement the optimal commitment solution. Blake and Kirsanova (2012) demonstrate the existence of multiple discretionary Markov-perfect equilibria in linear-quadratic models with rational expectations and endogenous state variables. They obtain unique equilibria in the absence of endogenous predetermined state variables. This does not contradict the implication of our approach that, in addition to the standard discretionary equilibrium, a discretionary equilibrium implementing the commitment solution exists in this case, as they consider Markov-perfect discretionary equilibria, whereas we consider discretionary equilibria where economic agents also respond to payoff-irrelevant state variables.

Alternatively, reputation-building can be modeled with the help of the sustainable equilibrium concept (Chari and Kehoe, 1990, 1993), which also relies on strategies that violate the Markov property. Compared to the sustainable-equilibrium concept, the discretionary equilibrium allows only for one-shot deviations by the policy-maker in each period rather than deviations which specify policies for all possible future histories. One advantage of the approach followed in this paper is that it highlights the parallels to the commitment solution and, in particular, that it allows for an intuitive interpretation of the additional state variables as Lagrange multipliers of the commitment solution.

The present analysis may also be reminiscent of the Folk theorem, which implies that for infinitely repeated games all individually rational payoff combinations can be achieved for sufficiently high discount factors (see Fudenberg and Maskin (1986) for one particular variant of the Folk theorem). Abreu et al. (1986, 1990) introduce a method to study reputation in infinitely repeated games that focuses on sets of continuation values as opposed to strategies. In contrast with these analyses, we do not consider infinitely repeated stage games, as the periods in our paper are connected via forward-looking and backward-looking variables.

This paper is organized as follows. In Section 2, we illustrate the main mechanism behind our result with the help of the canonical new Keynesian model. The implementability of the commitment solution by a discretionary policy-maker is demonstrated in Section 3 for a large class of linear-quadratic models with rational expectations. Section 4 concludes.

2 Simple New Keynesian Model

2.1 Framework

To show the main mechanism behind our general result, we first use the simple new Keynesian model as our workhorse (Clarida et al., 1999). In every period $t = 0, 1, 2, ...,$ the private sector's behavior is summarized by

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa u_t + \xi_t, \tag{1}$$

$$\xi_{t+1} = \rho \xi_t + \varepsilon_{t+1}, \tag{2}$$

where $\beta \in (0, 1)$ is a common discount factor, $\kappa > 0$, $\rho \in (0, 1)$, and the ε_t's are i.i.d. shocks that are drawn from a normal distribution with mean zero. π_t is the inflation rate and u_t the output gap. Equation (1) is the new Keynesian Phillips curve and (2) describes the evolution of markup shocks. The initial value of ξ_t, ξ_0, is exogenously given. Taking (1) and (2) into account, the central bank minimizes the expected discounted sum of losses

$$l(\pi_t, u_t) = \frac{1}{2}\pi_t^2 + \frac{a}{2}u_t^2 \tag{3}$$

with parameter a ($a > 0$). Adding a new Keynesian IS curve to the model would not affect our results. In our set-up, u_t constitutes the central bank's instrument. π_t is a forward-looking or non-predetermined variable. ξ_t is a predetermined variable.[5]

[5] The definitions of predetermined variables and non-predetermined variables are in line with Backus and Driffill (1986) and Söderlind (1999).

2.2 Optimal Commitment

We first construct the optimal commitment solution, which enables us to show later how a discretionary policy-maker can achieve this solution. The optimal commitment solution can be obtained by setting up the Lagrangian

$$\mathcal{L}_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2}\pi_t^2 + \frac{a}{2}u_t^2 + \lambda_{t+1} \left(\beta^{-1}\pi_t - \pi_{t+1} - \kappa\beta^{-1}u_t - \beta^{-1}\xi_t \right) \right), \tag{4}$$

where λ_t $(t = 1, 2, ...)$ are the multipliers associated with the new Keynesian Phillips curve.

As is well-known (Clarida et al., 1999; Woodford, 1999), the commitment solution can be characterized by

$$u_{t+1} - u_t = -\frac{\kappa}{a}\pi_{t+1} \qquad \textit{for t=0,1,2,...} \tag{5}$$

$$u_0 = -\frac{\kappa}{a}\pi_0 \tag{6}$$

together with (1) and (2).

In line with Backus and Driffill (1986) and Söderlind (1999), the commitment solution can be described by equations that specify the joint evolution of ξ_t and λ_t as well as by equations that state how the non-predetermined variable π_t and the instrument u_t depend on the current values of ξ_t and λ_t. With the help of the optimal commitment solution in Clarida et al. (1999), it is straightforward to show that, for the simple new Keynesian model under consideration, these equations are

$$\begin{pmatrix} \xi_{t+1} \\ \lambda_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ -\frac{\beta\delta}{1-\delta\beta\rho} & \delta \end{pmatrix} \begin{pmatrix} \xi_t \\ \lambda_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{t+1} \\ 0 \end{pmatrix}, \tag{7}$$

where

$$\delta = \frac{1 + \beta + \frac{\kappa^2}{a} - \sqrt{(1-\beta)^2 + 2(1+\beta)\frac{\kappa^2}{a} + \frac{\kappa^4}{a^2}}}{2\beta} \in (0, 1), \tag{8}$$

as well as

$$\begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} \frac{\delta}{1-\delta\beta\rho} & \frac{1-\delta}{\beta} \\ -\frac{\kappa\delta}{a(1-\delta\beta\rho)} & \frac{\kappa\delta}{a\beta} \end{pmatrix} \begin{pmatrix} \xi_t \\ \lambda_t \end{pmatrix}. \tag{9}$$

For exogenous values of ξ_0 and $\lambda_0 = 0$, (7) and (9) describe the entire dynamics of the system.

Loosely speaking, equation (7) can be interpreted as introducing a new predetermined state variable in addition to ξ_t: the Lagrange multiplier on the forward-looking constraint, λ_t. In the subsequent subsection, we show that a policy-maker acting under discretion may optimally respond to both variables ξ_t and λ_t in the same way as under commitment.

2.3 Discretionary Policy-Making

Abandoning the restriction to Markov-perfect strategies, we now construct a particular discretionary equilibrium, which implements the commitment solution. This equilibrium involves an additional state variable that, on the equilibrium path, equals the Lagrange multiplier in the commitment solution. While we consider this property of the additional state variable to be a desirable feature of our construction, as the additional state variable can be interpreted as the burden of "promises" made in the past, there are also discretionary equilibria implementing the commitment solution, for which this property does not hold.[6] It may also be noteworthy that additional non-Markovian discretionary equilibria exist in this framework that do not implement the commitment solution (see Hahn (2019)).

As a first step, we introduce a general law of motion for the additional state variable s_t. The dynamics of s_t are described by

$$s_{t+1} = \phi_\xi \xi_t + \phi_s s_t + \phi_u u_t \tag{10}$$

with a given initial value s_0. For payoff-relevant state variables, the coefficients in the law of motion are typically exogenously given. In the case under consideration, the ϕ's are arbitrary coefficients, which will be pinned down later. The flexibility with which the additional state variable can be introduced may be reminiscent of sunspots (see Cass and Shell (1983) for a seminal contribution). A major difference between the two concepts is the fact that sunspot variables are exogenous stochastic processes, whereas the state variable that we introduce is endogenous and can be influenced by the policy-maker, in particular.

[6] In particular, one can show that, for any $\zeta_0 \in \mathbb{R}$, $\zeta_1 \in \mathbb{R} \setminus \{0\}$, a non-Markovian discretionary equilibrium implementing the commitment solution exists with an additional payoff-irrelevant state variable that is equal to $\zeta_0 \xi_t + \zeta_1 \lambda_t$ on the equilibrium path.

We make use of this flexibility and set $s_0 = 0$ as well as

$$\phi_u = -\frac{\kappa}{1-\delta}, \tag{11}$$

$$\phi_\xi = -\frac{1}{1-\delta\beta\rho}, \tag{12}$$

$$\phi_s = \frac{1}{\beta}. \tag{13}$$

Why we choose these particular values will be explained in Section 2.4.

The following lemma implies that the commitment solution of the standard new Keynesian model can be implemented via a discretionary equilibrium:

Lemma 1. *Consider a central bank with instrument u_t, loss function (3), and discount factor β, who faces the constraints (1), (2), as well as the following law of motion for the payoff-irrelevant state variable s_t:*

$$s_{t+1} = \frac{1}{\beta}s_t - \frac{1}{1-\delta\beta\rho}\xi_t - \frac{\kappa}{1-\delta}u_t, \qquad \textit{with } s_0 = 0. \tag{14}$$

Then

$$\begin{pmatrix} \pi_t \\ u_t \end{pmatrix} = \begin{pmatrix} \frac{\delta}{1-\delta\beta\rho} & \frac{1-\delta}{\beta} \\ -\frac{\kappa\delta}{a(1-\delta\beta\rho)} & \frac{\kappa\delta}{a\beta} \end{pmatrix} \begin{pmatrix} \xi_t \\ s_t \end{pmatrix} \tag{15}$$

describes a discretionary equilibrium. This equilibrium implements the commitment solution.

Proof As a preliminary step, we note that the dynamics of π_t and u_t equal those in the commitment solution. This can be seen by comparing (14) and (15) to (7) and (9). In particular, plugging the expression for u_t given in (15) into (14) yields an expression for s_{t+1} as a function of ξ_t and s_t. With the help of (8) as well as $s_0 = \lambda_0 = 0$, it is straightforward to show that this equation for s_{t+1} yields dynamics for the additional state variable s_t that are identical to the dynamics for the Lagrange multiplier in the commitment solution, which are described by (7).

To show that (15) describes a discretionary equilibrium, we have to show that, first, together with (2) and (14), it is compatible with the Phillips curve (1) and that, second, the central bank's choice of u_t in every period t is optimal subject to the new Keynesian Phillips curve in this period t, given the rational inflation expectations $\mathbb{E}_t\pi_{t+1}$ that are formed in line with (2), (14), and (15).

The candidate equilibrium is compatible with the new Keynesian Phillips, because (i) the economy evolves as in the commitment solution and (ii) the commitment solution satisfies the new Keynesian Phillips curve. To show that the central bank behaves optimally requires a few steps. First, we calculate inflation expectations as they enter the new Keynesian Phillips curve, which represents a constraint for the central bank. Using (14) and (15), we obtain the following expression:

$$\mathbb{E}_t \pi_{t+1} = \frac{1}{1-\delta\beta\rho}\left(\delta\rho - \frac{1-\delta}{\beta}\right)\xi_t + \frac{1-\delta}{\beta^2}s_t - \frac{\kappa}{\beta}u_t. \tag{16}$$

As we will see later, it is crucial that inflation expectations are a function of u_t. This feature stems from the fact that the central bank's choice of u_t affects s_{t+1}, which in turn affects inflation in period $t+1$.

As a next step, we plug the inflation expectations (16) into the new Keynesian Phillips curve (1) and obtain:

$$\pi_t = \frac{\delta}{1-\delta\beta\rho}\xi_t + \frac{1-\delta}{\beta}s_t. \tag{17}$$

It is noteworthy that the central bank's choice of u_t, has no influence on inflation in the same period. This has been achieved by a particular choice of ϕ_u, as will be explained in Section 2.4.

Consider optimal central-bank behavior in a particular period t. For this purpose, we set up the corresponding Bellman equation:

$$\begin{gathered} W(\xi_t, s_t) = \min_{u_t}\left\{\frac{1}{2}\pi_t^2 + \frac{a}{2}u_t^2 + \beta\mathbb{E}_t W(\xi_{t+1}, s_{t+1})\right\} \\ \textit{subject to (2), (17).} \end{gathered} \tag{18}$$

We obtain the following first-order condition as well as a condition that results from the envelope theorem:

$$0 = au_t - \frac{\kappa}{1-\delta}\beta\mathbb{E}_t W_s(\xi_{t+1}, s_{t+1}), \tag{19}$$

$$W_s(\xi_t, s_t) = \frac{1-\delta}{\beta}\pi_t + \mathbb{E}_t W_s(\xi_{t+1}, s_{t+1}) \tag{20}$$

Equations (19) and (20) can be combined to

$$\mathbb{E}_t u_{t+1} - u_t = -\frac{\kappa}{a}\mathbb{E}_t \pi_{t+1} \qquad \textit{for t=0,1,2,...} \tag{21}$$

This condition characterizes optimal central-bank behavior in the candidate discretionary equilibrium. As the paths of π_t and u_t satisfy the condition for optimal central-bank behavior under commitment, (5), they also satisfy (21). □

It may be noteworthy that, in the discretionary equilibrium under consideration, there is no separate condition for period 0 as opposed to the commitment solution, which involves a special condition for this period, equation (6). However, it is clear that (6) is satisfied also in the discretionary equilibrium.

One might ask whether the discretionary equilibrium specified in Lemma 1 is the only equilibrium for the new Keynesian model with the additional state variable s_t, introduced by (14). This is not the case. The standard Markov-perfect discretionary equilibrium would also correspond to an equilibrium of the economy with the additional state variable s_t. In this equilibrium, u_t and π_t would not be affected by the value of s_t. Söderlind (1999) presents matlab routines that can be used to compute discretionary equilibria and optimal solutions under commitment. Experiments with the simple new Keynesian model with additional state variable s_t suggest that the algorithm for discretionary policy-making converges to the additional discretionary equilibrium that implements the commitment solution, provided that starting values close to the respective equilibrium values are used. In other cases, the algorithm converges to the equilibrium where the additional state variable s_t does not affect the dynamics of π_t and u_t.

Finally, one might wonder whether a discretionary policy-maker could also implement the policy that is optimal from a timeless perspective (Woodford, 1999). The timeless-perspective solution is identical to the commitment solution but imposes (5) rather than (6) also in the initial period. A discretionary policy-maker could easily implement this solution for the appropriate initial value s_0.

2.4 Interpretation

It may be instructive to explain how the parameter choices for the law of motion for s_t were made and why these choices enable the commitment solution to be implemented. In the Markov-perfect equilibrium of the standard new Keynesian model, there is no endogenous state variable, and inflation in all periods is only a function of the exogenous state variable ξ_t. Hence inflation expectations $\mathbb{E}_t \pi_{t+1}$ cannot be influenced by the central bank's choice of u_t, as ξ_{t+1} is exogenous to monetary policy. Together with this observation, the Phillips curve (1) implies that an increase in u_t affects inflation only via the traditional marginal-cost channel, where an increase of u_t by Δu entails an increase in inflation by $\kappa \Delta u$.

In a discretionary equilibrium that violates the Markov property, changes in the instrument u_t may lead to changes in inflation expectations because changes in u_t influence an additional, endogenous state variable, which, in turn, affects inflation expectations. Thus the central bank can affect current inflation π_t not only via the traditional marginal cost channel, which we have described above, but also via an expectations channel. The particular choice of ϕ_u in (11) guarantees that the effects of a change in u_t on inflation via the marginal-cost channel and the expectations channel exactly offset each other. As a consequence, inflation π_t has effectively become a predetermined variable, which cannot be influenced by the central bank's choice in period t (see (17)).

This observation is key to understanding how the discretionary equilibrium under consideration implements the commitment solution. Intuitively, a discretionary policy-maker cannot implement the commitment solution in a Markov-perfect equilibrium of the simple new Keynesian model because, in every period t, it cannot make a binding commitment about inflation in period $t+1$. In period $t+1$, the policy-maker will always succumb to the temptation to select the inflation rate that is optimal from the perspective of period $t+1$. In the non-Markov-perfect equilibrium that we have constructed, the central bank does not renege on past promises because it cannot do so.

The parameter choices for ϕ_ξ and ϕ_s were made in a way such that the evolution of the variables π_t, u_t, and s_t equals the evolution of the corresponding variables π_t, u_t, and λ_t in the commitment solution. The choice of $s_0 = 0$ ensures $\lambda_0 = s_0$.

2.5 Dynamic Response to Policy Errors

We continue our discussion of the standard new Keynesian model by illustrating the dynamics of the economy if the central bank deviates from its optimal choice of u_t in period 0 but not in future periods. We set the parameters as follows: $\kappa = 0.3$, $\beta = 0.99$, $\rho = 0.9$, $a = 0.05$, and $\xi_0 = 0$, in addition to $s_0 = 0$. Moreover, assume that the central bank chooses $u_0 = 1$ rather than $u_0 = 0$, which would be optimal.

Figure 1 illustrates the dynamics of the system in the absence of markup shocks ξ_t, i.e. for $\varepsilon_t = 0$ in $t = 1, 2, 3, ...$ The deviation of the central bank cannot influence inflation (displayed as a solid line) in period 0, as the discretionary equilibrium under consideration involves that inflation is effectively predetermined. According to the

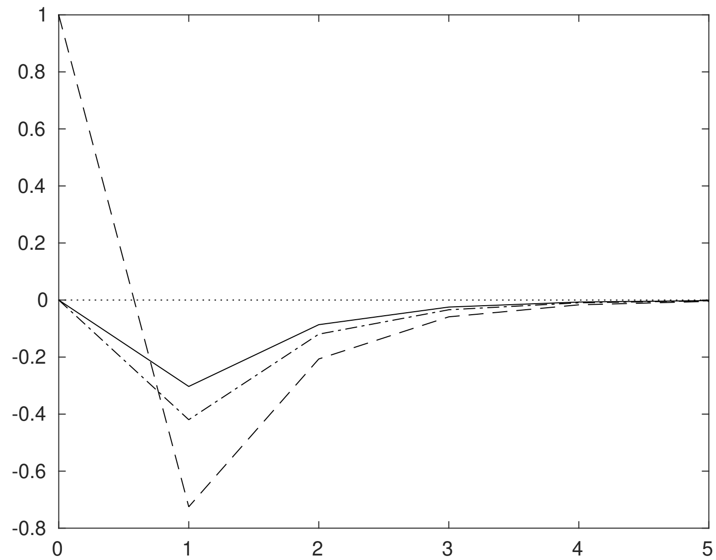

Figure 1: Impulse responses of inflation (solid line), the output gap (dashed line), and the additional state variable s_t (dash-dotted line) in response to a one-time deviation of the output gap. Markup shocks have been set to zero in all periods.

dash-dotted line, which displays s_t, the suboptimally high level of u_0 drives s_1 below zero (compare (10) and (11)). Because s_{t+1} depends positively on s_t on the equilibrium path ($s_{t+1} = \delta s_t$), s_t stays negative in the consecutive periods $t = 2, 3, ...$ As inflation and output are increasing functions of s_t (compare (9) for $s_t = \lambda_t$), both output and inflation are negative from period 1 onward.

The figure shows that deviations from the equilibrium behavior in a particular period lead to changes in the additional state variable s in future periods and thereby to changes in losses in these periods. It may be interesting to contrast the consequences of a deviation in this paper and the respective consequences in papers that use trigger strategies to overcome time-inconsistency problems (see, e.g., Loisel, 2008; Levine et al., 2008).[7] Compared to equilibria with trigger strategies, a small deviation of the central bank only has small, transitory consequences for the economy. As can be seen easily, the response of the economy after a deviation is always proportional to the magnitude of the deviation.

It may also be instructive to consider the situation where s_t differs from zero in one period t, there are no markup shocks, i.e. $\xi_t = 0$, and the central bank ignores the burden of past promises, s_t, by setting $u_t = 0$ in this period. In this case, $s_{t+1} = \beta^{-1} s_t$

[7] Abreu et al. (1990) show that, under certain conditions, efficient sequential equilibria of infinitely repeated games have the so-called bang-bang property, which involves that only extreme payoff combinations occur for all possible histories.

would hold. Thus the burden from past "promises" would grow by a factor β^{-1}. This mild increase in s_t deters deviations in the first place.

2.6 Bounded errors

As has been highlighted in the previous analysis, one advantage of our approach is that the punishment after a deviation is always in proportion to the deviation itself. To see this from a slightly different angle, assume that the shocks ε_t are drawn from a distribution with bounded support $[-\bar{\varepsilon}, +\bar{\varepsilon}]$. It is then easy to see that the state variables ξ_t and s_t would remain bounded in equilibrium. In particular, ξ_t would lie in the interval $\left[-\frac{1}{1-\rho}\bar{\varepsilon}, -\frac{1}{1-\rho}\bar{\varepsilon}\right]$ in all periods, provided that this is true for the initial value ξ_0. As a consequence, the policy-maker's choices of instrument u_t would also be bounded in all periods. As is straightforward to show, this would also be true for all future periods if the policy-maker deviated to the standard (Markovian) discretionary solution in a particular period.[8] Moreover, inflation would remain bounded as well. The boundedness of π_t and u_t is an immediate consequence of the observation that the state variable s_t remains in a bounded set in all periods following a deviation to the choice that would be optimal in the standard Markovian discretionary equilibrium. In this sense, our approach always relies only on moderate punishments.

3 General Result

3.1 Set-up

In the following, we generalize the results for the simple new Keynesian model to a general class of linear-quadratic models with rational expectations. There are n_x predetermined variables, contained in the column vector x_t, and n_y non-predetermined variables, contained in the vector y_t. Let z_t be the $(n_x + n_y)$-dimensional vector that contains all predetermined and all non-predetermined variables, where the predetermined variables come first. There is also a k-dimensional vector of instruments u_t.

The policy-maker's objectives are described by

$$L = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(z_t' Q z_t + 2 z_t' U u_t + u_t' R u_t \right), \tag{22}$$

[8]The standard Markovian discretionary solution involves $u_t = -\frac{\kappa}{\kappa^2 + a(1-\beta\rho)} \xi_t$.

where $\beta \in (0,1)$. Q is an $(n_x+n_y)\times(n_x+n_y)$-dimensional matrix, U is an $(n_x+n_y)\times k$ dimensional matrix, and R is a $k \times k$-dimensional matrix. Without loss of generality, we assume Q and R to be symmetric.

The predetermined and non-predetermined variables evolve according to

$$x_{t+1} = A_{xx}x_t + A_{xy}y_t + B_x u_t + \varepsilon_{x,t+1}, \quad (23)$$

$$\mathbb{E}_t y_{t+1} = A_{yx}x_t + A_{yy}y_t + B_y u_t, \quad (24)$$

where A_{xx}, A_{xy}, A_{yx}, A_{yy}, B_x, and B_y are given matrices whose coefficients have been obtained from log-linearized equations describing the private-sector equilibrium, for example. The n_x components of $\varepsilon_{x,t+1}$ describe the innovations to the predetermined variables x_{t+1}. They have zero mean and covariance matrix Σ.

As is well-known, models with lagged variables and expectations more than one period ahead can also be cast in the form considered here. The same is also be true for some models with lagged expectations of present and future variables (see Blanchard and Kahn (1980)).

3.2 Commitment

The commitment solution can be obtained by setting up the Lagrangian (Backus and Driffill, 1986; Söderlind, 1999)

$$\mathcal{L}_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \left(z_t' Q z_t + z_t' U u_t + u_t' R u_t + \rho_{t+1}' \left(A z_t + B u_t - z_{t+1} \right) \right), \quad (25)$$

where

$$A = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix}, B = \begin{pmatrix} B_x \\ B_y \end{pmatrix}. \quad (26)$$

ρ_t is an $(n_x + n_y)$-dimensional vector of Lagrange multipliers $(t = 1, 2, 3, ...)$. The first-order conditions with respect to z_t and u_t are

$$\beta A' \mathbb{E}_t \rho_{t+1} = -\beta Q z_t - \beta U u_t + \rho_t, \quad (27)$$

$$-B' \mathbb{E}_t \rho_{t+1} = U' z_t + R u_t. \quad (28)$$

We assume that the commitment solution exists and involves unique paths of z_t and u_t. As shown by Backus and Driffill (1986), the commitment solution can then be described by an equation that specifies the evolution of the predetermined variables x_t and the multipliers $\rho_{y,t}$ associated with the non-predetermined variables

$$\begin{pmatrix} x_{t+1} \\ \rho_{y,t+1} \end{pmatrix} = H \begin{pmatrix} x_t \\ \rho_{y,t} \end{pmatrix} + \begin{pmatrix} \varepsilon_{t+1} \\ 0 \end{pmatrix}, \tag{29}$$

where H is an $(n_x + n_y) \times (n_x + n_y)$-dimensional matrix. The initial values of the predetermined variables x_t are exogenously given. The initial values of the elements in $\rho_{y,t}$ are zero because the non-predetermined variables can be chosen freely in $t = 0$. The non-predetermined variables y_t, the Lagrange multipliers $\rho_{x,t}$ associated with the predetermined variables, and the policy-makers' instruments u_t can be expressed as functions of the current values of x_t and $\rho_{y,t}$. In particular, y_t can be written as

$$y_t = C \begin{pmatrix} x_t \\ \rho_{y,t} \end{pmatrix}. \tag{30}$$

C is an $n_y \times (n_x + n_y)$-dimensional matrix that can be decomposed as $C = \left(C_x, C_{\rho_y}\right)$, where C_x has dimensions $n_y \times n_x$ and C_{ρ_y} has size $n_y \times n_y$.

3.3 Discretionary Policy-Making

We are now in a position to formulate our main result:

Proposition 1. *Consider the unique commitment solution of the model characterized by (23), (24) and loss function (22). Let $C = (C_x, C_{\rho_y})$ be the matrix that describes how y_t depends on x_t as well as $\rho_{y,t}$ (see (30)). Then C_{ρ_y} is invertible. For the discretionary policy-maker, introduce an n_y-dimensional vector of additional state variables, s_t. In the initial period, set $s_0 = 0_{n_y}$.*[9] *For $t = 0, 1, 2, \ldots$, assume that s_t evolves according to*

$$s_{t+1} = A_{sx} x_t + A_{ss} s_t + B_s u_t, \tag{31}$$

where the matrices B_s, A_{sx}, and A_{ss} are given by

$$\begin{aligned} B_s &= C_{\rho_y}^{-1} \left(B_y - C_x B_x\right), & (32)\\ A_{sx} &= C_{\rho_y}^{-1} \left[\left(A_{yy} - C_x A_{xy}\right) C_x - C_x A_{xx} + A_{yx}\right], & (33)\\ A_{ss} &= C_{\rho_y}^{-1} \left(A_{yy} - C_x A_{xy}\right) C_{\rho_y}. & (34) \end{aligned}$$

[9] We use the definition that 0_{n_y} is an n_y-dimensional column vector of zeros.

Then a discretionary equilibrium for (23), (24), (31) and loss function (22) exists that implements the commitment solution.

The proof is given in Appendix A.

The main idea for the proof of the general result is similar to the one for the new Keynesian model. First, the fact that the commitment solution can be described by the joint dynamics of x_t and $\rho_{y,t}$ suggests that n_y additional state variables should be added to the discretionary policy-maker's problem in addition to the payoff-relevant state variables x_t. These additional state variables are the components of s_t. Second, to ensure that the policy-maker does not renege on past promises, the dynamics of s_t are specified in a way such that, in every period t, the policy-maker cannot influence current non-predetermined variables (as opposed to future non-predetermined variables). This is achieved by (32), which is a straightforward generalization to (11). Third, (33) and (34) specify the matrices A_{ss} and A_{sx} in a manner such that the dynamics of s_t on the equilibrium path of the discretionary equilibrium equal those of $\rho_{y,t}$ in the commitment solution. Finally, one needs to show that the discretionary policy-maker does not wish to deviate. At this point of the proof, it is helpful that the value function of the discretionary policy-maker can be constructed from the commitment solution under the assumption that both approaches lead to identical dynamics.

We stress that the equilibrium for the discretionary policy-maker's problem specified in Proposition 1 is typically not unique.[10] In all cases where the economy admits a Markov-perfect discretionary equilibrium, this equilibrium will also correspond to an equilibrium for the economy where the additional state variables s_t have been added via (31).

It may also be interesting to relate Proposition 1 to a theorem in Backus and Driffill (1986) that examines whether the commitment solution can be supported by trigger strategies.[11] In particular, they consider the case where deviations from the commitment solution are punished by a grim trigger, i.e. a permanent switch to the standard Markov-perfect discretionary equilibrium. According to their theorem, the commitment solution can be sustained (i) if the support of shocks is bounded and (ii) if the discount factor is sufficiently large. By contrast, the discretionary equilibria constructed in this paper always allow for the implementation of the commitment solution, irrespective of the magnitude of the discount factor and for an arbitrary support

[10] See also our related discussion in Section 3.3.

[11] Currie and Levine (1993) find an analogous result for continuous-time models. A related finding for the new Keynesian model is due to Kurozumi (2008).

of shocks. This result stems from the fact that, loosely speaking, the punishment for deviations from the commitment solution is not restricted to a switch to the standard discretionary equilibrium.

4 Conclusions

This paper has shown that the implementation of the commitment solution is often possible also for policy-makers who lack a means of commitment. As a consequence, the main task for the policy-maker is to help coordinate economic agents on the equilibrium that facilitates the commitment outcome.

An interesting avenue for future research would be whether solutions under intermediate degrees of commitment (Schaumburg and Tambalotti, 2007; Debortoli and Nunes, 2010; Debortoli et al., 2014), where the policy-maker may renege on past promises with a constant probability every period, can be implemented by a fully discretionary policy-maker as well. It appears plausible that this is possible in a variant of our set-up where, in every period, all additional payoff-irrelevant state variables are set to zero with a constant exogenous probability. In addition, it may be instructive to examine variants of our approach where these state variables are not set to zero completely but where their magnitude may be reduced from time to time. This would capture a mild form of loose commitment where past commitments are not lost completely.

A Proof of Proposition 1

It will be useful to note that the commitment solution can also be obtained by an alternative approach outlined in Backus and Driffill (1986). They formulate (23) and (24) jointly as

$$z_{t+1} = Az_t + Bu_t + \varepsilon_{t+1}, \tag{35}$$

where the first n_x elements of ε_{t+1} are the exogenous disturbances $\varepsilon_{x,t+1}$ and the remaining components are given by the endogenous expectational errors $\varepsilon_{y,t+1}$, which have to satisfy the requirement that $\mathbb{E}_t\varepsilon_{y,t+1} = 0_{n_y}$ for $t = 0, 1, 2, ...$

In a first step of the alternative approach, the policy-maker takes z_t as given when it makes its choice regarding u_t for $t = 0, 1, 2, ...$ In a second step, it chooses the initial value of the non-predetermined variable, y_0, as well as the expectational errors $\varepsilon_{y,t}$ for $t = 1, 2, 3, ...$

Thus in the first step, the policy-maker selects u_t to minimize

$$\begin{gathered}\min_{u_t} \left\{ z_t'Qz_t + 2z_t'Uu_t + u_t'Ru_t + \beta\mathbb{E}_t\left[z_{t+1}'Vz_{t+1} + k\right]\right\}, \\ \textit{subject to (35), } z_t \textit{ given,}\end{gathered} \tag{36}$$

where $z_{t+1}'Vz_{t+1} + k$ is the cost-to-go at $t + 1$ with a $(n_x + n_y) \times (n_x + n_y)$-dimensional symmetric matrix V and a constant k. Importantly, V is related to the multipliers from the Lagrangian approach via[12]

$$\rho_t = \beta V z_t. \tag{37}$$

We construct matrices V_{xx}, V_{xy}, V_{yx}, and V_{yy} by partitioning V conformably with x_t and y_t.

In the second step, the policy-maker selects y_0 as well as the endogenous forecast errors $\varepsilon_{y,t+1}$. The latter choice is immaterial for our purposes and therefore omitted. The former choice is obtained as a result of minimizing $z_0'Vz_0 = x_0'V_{xx}x_0 + 2x_0'V_{xy}y_0 + y_0'V_{yy}y_0$ with respect to y_0 for given x_0. The corresponding first-order condition is

$$V_{yx}x_0 + V_{yy}y_0 = 0. \tag{38}$$

[12] There appears to be a small mistake in Backus and Driffill (1986), as they omit the discount factor in the relationship between ρ_t and Vz_t. That (37) is correct can be confirmed by comparing (28) and the first-order condition for optimization problem (36), which is $U'z_t + Ru_t + \beta B'V\mathbb{E}_t z_{t+1} = 0$.

Since we have assumed that the commitment solution involves a unique path of all economic variables, the optimal choice of y_0 is unique. Thus we can conclude that V_{yy} is invertible.

Comparing (30) and $\rho_{y,t} = \beta V_{yx} x_t + \beta V_{yy} y_t$ (see (37)) leads to the implication that C_{ρ_y} is invertible as well and that

$$\begin{aligned} V_{yx} &= -\beta^{-1} C_{\rho_y}^{-1} C_x, \quad &(39)\\ V_{yy} &= \beta^{-1} C_{\rho_y}^{-1}. \quad &(40) \end{aligned}$$

It will be useful to introduce matrix T as

$$T = \begin{pmatrix} I_{n_x} & 0_{n_x \times n_y} \\ \beta V_{yx} & \beta V_{yy} \end{pmatrix}, \quad (41)$$

where I_{n_x} is the $n_x \times n_x$-dimensional identity matrix and $0_{n_x \times n_y}$ is an $n_x \times n_y$ matrix of zeros. Matrix T allows us to transform $z_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ into $\begin{pmatrix} x_t \\ \rho_{y,t} \end{pmatrix}$:

$$\begin{pmatrix} x_t \\ \rho_{y,t} \end{pmatrix} = T \begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad (42)$$

We note that T is invertible with

$$T^{-1} = \begin{pmatrix} I_{n_x} & 0_{n_x \times n_y} \\ -V_{yy}^{-1} V_{yx} & \beta^{-1} V_{yy}^{-1} \end{pmatrix}. \quad (43)$$

After these preliminary steps, we now begin to formulate the optimization problem under discretion, assuming that all variables z_t and u_t evolve as in the commitment solution and introducing a vector of additional state variables s_t, which is always identical to $\rho_{y,t}$ on the equilibrium path. We postulate that, in period $t+1$, y_{t+1}, will depend on x_{t+1} and s_{t+1} in the same way that y_{t+1} depends on x_{t+1} and $\rho_{y,t+1}$ in the commitment solution (compare (30)). In this case, (24) can be formulated as

$$\mathbb{E}_t C \begin{pmatrix} x_{t+1} \\ s_{t+1} \end{pmatrix} = A_{yx} x_t + A_{yy} y_t + B_y u_t. \quad (44)$$

Combining (23), (31), and (44) yields

$$(A_{yy} - C_x A_{xy})\, y_t = \left(C_x A_{xx} + C_{\rho_y} A_{sx} - A_{yx}\right) x_t + C_{\rho_y} A_{ss} s_t \\ + \left(C_x B_x + C_{\rho_y} B_s - B_y\right) u_t. \tag{45}$$

We choose B_s such that $C_x B_x + C_{\rho_y} B_s - B_y = 0$, which is equivalent to (32). This entails that the policy-maker cannot affect y_t by changing u_t:

$$(A_{yy} - C_x A_{xy})\, y_t = \left(C_x A_{xx} + C_{\rho_y} A_{sx} - A_{yx}\right) x_t + C_{\rho_y} A_{ss} s_t \tag{46}$$

Comparing with (30) yields

$$(A_{yy} - C_x A_{xy})\, C_x = C_x A_{xx} + C_{\rho_y} A_{sx} - A_{yx}, \tag{47}$$

$$(A_{yy} - C_x A_{xy})\, C_{\rho_y} = C_{\rho_y} A_{ss}. \tag{48}$$

These equations are equivalent to (33) and (34). They pin down A_{sx} and A_{ss}.

We introduce a symmetric $(n_x + n_y) \times (n_x + n_y)$-dimensional matrix W such that, conditional on optimal behavior by the policy-maker in every period, the present value of discounted losses is $(x_t', s_t')W \begin{pmatrix} x_t \\ s_t \end{pmatrix}$, up to a constant term. The discretionary policy-maker's optimization problem can then be stated as

$$\min_{u_t} \left\{ 2\begin{pmatrix} x_t' & y_t' \end{pmatrix} U u_t + u_t' R u_t + \beta \mathbb{E}_t \left[(x_{t+1}', s_{t+1}') W \begin{pmatrix} x_{t+1} \\ s_{t+1} \end{pmatrix} \right] \right\} \tag{49}$$

subject to (23), (31), x_t, y_t, *and* s_t *given.*

It is noteworthy that the term $z_t' Q z_t$ from the loss function can be ignored in the minimization problem because x_t is predetermined in t and y_t, due to our choice of B_s, is effectively predetermined as well.

As a next step, we note that the matrix W is related to V via the relation

$$W = \left(T^{-1}\right)' V T^{-1} \tag{50}$$

because the candidate discretionary equilibrium we are constructing involves the same cost-to-go as the commitment solution.

With the help of (50), we can state the first-order condition for optimization problem (49) as

$$U'z_t + Ru_t + \beta \begin{pmatrix} B_x' & B_s' \end{pmatrix} \left(T^{-1}\right)' V T^{-1} \mathbb{E}_t \begin{pmatrix} x_{t+1} \\ s_{t+1} \end{pmatrix} = 0. \tag{51}$$

Because of

$$T^{-1} \begin{pmatrix} x_{t+1} \\ s_{t+1} \end{pmatrix} = \begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = z_{t+1}, \tag{52}$$

$$\beta V \mathbb{E}_t z_{t+1} = \mathbb{E}_t \rho_{t+1}, \tag{53}$$

and

$$\begin{pmatrix} B_x' & B_s' \end{pmatrix} \left(T^{-1}\right)' = \begin{pmatrix} B_x' & B_y' \end{pmatrix} = B', \tag{54}$$

the condition for optimal behavior by the discretionary policy-maker, (51), is equivalent to (28), which holds for the commitment solution. Thus we have constructed a discretionary equilibrium that implements the commitment solution. □

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