Game Theory

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Mixed Strategy Nash Equilibrium
Mixed Strategy

- Until now, we only consider pure strategies, where players make their choices with certainty.
- von Neumann-Morgenstern preferences have degenerated probability distribution of 0 and 1
- However, in a lot of cases, there may be uncertainty about strategies played by other players
  - Players may not be able to exactly anticipate which strategies the other players play.
  - Players could also randomize when faced with a choice.
  - Strategies may be heterogeneous in the population
- We extend the strategic game by allowing each player choosing strategy profiles with probabilities $p = [0, 1] \Rightarrow$ Mixed strategies
- vNM-expected utility: $EU = p_1 U(s_1) + ... + p_n U(s_n)$
Strategic game with vNM preferences

A strategic game with vNM preferences consists of

- a set of players $N = 1, ..., n$ with element $i \in N$
- For each player $i$ there is a set of strategies $S_i$ with element $s_i \in S_i$
- For each player, preferences over uncertain strategy profiles are represented by the **expected value of a payoff function** over strategy profiles.

- Consider again the prisoners’ dilemma:

<table>
<thead>
<tr>
<th></th>
<th>Q</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q</td>
<td>2,2</td>
<td>0,3</td>
</tr>
<tr>
<td>F</td>
<td>3,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

- Assume, player 1 believes that player 2 will play $Q$ with a probability of $\sigma = 0.5$ and play $F$ with probability $1 - \sigma = 0.5$

- The expected payoff of player 1 of playing $F$ is $\sigma u_1(F, Q) + (1 - \sigma) u_1(F, F) = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 1 = 2$
Mixed strategy

- A mixed strategy allows players not only to restrict their choices to a single deterministic action, but to choose a probability distribution over her set of strategies.

- We denote such a probability distribution a **mixed strategy**, i.e. a list of probabilities in the order of strategies.

**Mixed strategy**

A mixed strategy of a player is a probability distribution over the set of his strategies. If player $i$ has strategies $S_i = \{s_{i1}, s_{i2}, ..., s_{ik}\}$ then a mixed strategy for player $i$ assigns to each pure strategy $s_i \in S_i$ a probability $0 \leq \sigma_i(s_i) \leq 1$ that will be played, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$

- Prisoners’ dilemma example: The mixed strategy $(\frac{1}{2}, \frac{1}{2})$ for player 2 assigns probability $\frac{1}{2}$ to $Q$ and probability $\frac{1}{2}$ to $F$

- A mixed strategy may also assign a probability of 1 to a strategy → **pure strategy**
Example: Calculate expected payoff

<table>
<thead>
<tr>
<th>$s_1 / s_2$</th>
<th>L</th>
<th>R</th>
<th>$\sigma_2(\sigma_{2L}, \sigma_{2R})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,3</td>
<td>3,4</td>
<td>$\frac{11}{4}, \ldots$</td>
</tr>
<tr>
<td>D</td>
<td>1,2</td>
<td>6,0</td>
<td>\ldots,\ldots</td>
</tr>
</tbody>
</table>

$\sigma_1(\sigma_{1U}, \sigma_{1D}) = (\frac{1}{3}, \frac{2}{3})$

|          | $\frac{7}{3}$ | \ldots,\ldots | \ldots,\ldots |

- Mixed strategy for player 1: $\sigma_1(\frac{1}{3}, \frac{2}{3})$
- Mixed strategy for player 2: $\sigma_2(\frac{1}{4}, \frac{3}{4})$
- Profile of mixed strategies $\sigma = (\sigma_1, \sigma_2)$
- Expected payoff for player 1 when playing $U$ is $\frac{1}{4} \cdot 2 + \frac{3}{4} \cdot 3 = \frac{11}{4}$
- Expected payoff for player 2 when playing $L$ is $\frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 2 = \frac{7}{3}$
A mixed strategy Nash equilibrium is a mixed strategy profile $\sigma^* = (\sigma^*_i, \sigma^*_{-i})$ with the property that no player $i$ has a mixed strategy $\sigma_i$ such that she prefers the outcome of the profile $\sigma = (\sigma_i, \sigma^*_{-i})$ over the outcome of the strategy profile $\sigma = (\sigma^*_i, \sigma^*_{-i})$.

For every $i \in N$, $U_i(\sigma^*_i, \sigma^*_{-i}) \geq U_i(\sigma_i, \sigma^*_{-i}) \forall \sigma_i \in \Delta(S_i)$, where $\Delta(S_i)$ is the set of probability distribution over $S_i$ with typical element $\sigma_i$.

A mixed-strategy profile $\sigma^* = (\sigma^*_1, ..., \sigma^*_n)$ is a Nash equilibrium if and only if for every player $i$, $\sigma^*_i$ is a best response to $\sigma^*_{-i}$.
Finding Mixed strategy Nash equilibria

- In large games, it is not easy to find mixed-strategy Nash equilibria.

- In games with 2 players and 2 pure strategies, it is quite easy and can also be represented graphically.

- In 2x2 games, a mixed strategy consists of two real numbers that sum up to 1 → the mixed strategy is represented by a single real number in the interval [0, 1].

- A profile of mixed strategies for two players can then be represented by two real numbers in the interval [0, 1].

- 2 real numbers in this interval can be represented as a point in the unit square.
Consider again our example with Marge (player 1) and Homer (player 2):

<table>
<thead>
<tr>
<th>Marge/Homer</th>
<th>theater</th>
<th>TV</th>
</tr>
</thead>
<tbody>
<tr>
<td>theater</td>
<td>5,1</td>
<td>0,0</td>
</tr>
<tr>
<td>TV</td>
<td>0,0</td>
<td>1,5</td>
</tr>
</tbody>
</table>

We have identified 2 pure strategy Nash equilibria.

Both have 2 pure strategies available: \( S_1 = (\text{theater}, \text{TV}) \) and \( S_2 = (\text{theater}, \text{TV}) \)

A mixed strategy for Marge would be: \( \sigma_1(\sigma_{1,\text{theater}}, \sigma_{1,\text{TV}}) \) is represented by a real number in \([0, 1]\) and is denoted \( p \rightarrow \) hence, \( \sigma_{1,\text{theater}} = 1 - p \)

Equivalently, a mixed strategy for Homer is: \( \sigma_2(\sigma_{2,\text{theater}}, \sigma_{2,\text{TV}}) \) is represented by a real number in \([0, 1]\) and is denoted \( q \rightarrow \) hence, \( \sigma_{2,\text{theater}} = 1 - q \)
Example (cont.)

- The profile of mixed strategy $\sigma(\sigma_1, \sigma_2)$ can then be represented by a vector in $[0, 1] \times [0, 1]$

- Corner points are pure strategies
- How can we easily find a mixed strategy equilibrium? → Look at best response functions
Example (cont.)

- Suppose, Marge believes that Homer will watch TV with probability $q$

- Marge’s expected payoff when she chooses theater is:
  \[ u_{Marge}(p = 0, q) = 5 \cdot (1 - q) + 0 \cdot (q) = 5 - 5 \cdot q \]

- Marge’s expected payoff when she chooses TV is:
  \[ u_{Marge}(p = 1, q) = 0 \cdot (1 - q) + 1 \cdot (q) = q \]

- It is best for Marge to choose theater (by choosing $p = 0$) if $q < \frac{5}{6}$

- Equivalently, it is best for Marge to choose TV (by choosing $p = 1$) if $q > \frac{5}{6}$

- If $q = \frac{5}{6}$?
What is the mixed-strategy best response of Marge?

<table>
<thead>
<tr>
<th></th>
<th>(1 − q)</th>
<th>q</th>
</tr>
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<tr>
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<td>5,1</td>
<td>0,0</td>
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Marge’s expected payoff from playing $p$, if Homer plays $q$:

$$u_{Marge}(p, q) = 5 \cdot (1 − p) \cdot (1 − q) + 0 \cdot (1 − p) \cdot q + 0 \cdot p \cdot (1 − q) + 1 \cdot p \cdot q =$$

$$= 5 − 5q + p \left(6q − 5\right)$$

Note: here we do not only consider pure strategies (as on the previous slide), but also every possible mixed strategy.
Marge’s payoff is increasing in $p$ if $6q > 5$ and decreasing in $p$ if $6q < 5$

so, her best response to choose $p$ is $B_{\text{Marge}}(q) = \{1\}$ for $q > \frac{5}{6}$ and $\{0\}$ for $q < \frac{5}{6}$

She is indifferent among all her mixed strategies when $q = \frac{5}{6}$

Hence, Marge’s best response function is

$$B_{\text{Marge}}(q) = \begin{cases} 
\{0\} & \text{if } q \in [0, \frac{5}{6}) \\
[0, 1] & \text{if } q = \frac{5}{6} \\
\{1\} & \text{if } q \in (\frac{5}{6}, 1] 
\end{cases}$$
We can draw Marge’s best response function:
Example (cont.)

- We can do the same exercise for Homer and get the following best response function:

\[
B_{Homer}(p) = \begin{cases} 
\{0\} & \text{if } p \in \left[0, \frac{1}{6}\right) \\
[0, 1] & \text{if } p = \frac{1}{6} \\
\{1\} & \text{if } p \in \left(\frac{1}{6}, 1\right]
\end{cases}
\]

- Nash equilibria occur, where best responses cross
We can draw Homer’s best response function:
3 Nash equilibria: 2 pure strategy equilibria and 1 mixed strategy equilibrium

The Nash equilibria for this game are:

\[ \sigma^* = (\sigma_1^*, \sigma_2^*) = [p^*, q^*] = \{(0, 0); (1, 1); (\frac{1}{6}, \frac{5}{6})\} \]

Generically, games have odd numbers of equilibria
Consider the expected profits of the game:

<table>
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<tr>
<th></th>
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<th>TV</th>
<th>$\sigma_2 = (\frac{1}{6}, \frac{5}{6})$</th>
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<tbody>
<tr>
<td>theater</td>
<td>5,1</td>
<td>0,0</td>
<td>$\frac{5}{6}, \frac{1}{6}$</td>
</tr>
<tr>
<td>TV</td>
<td>0,0</td>
<td>1,5</td>
<td>$\frac{5}{6}, \frac{25}{6}$</td>
</tr>
<tr>
<td>$\sigma_1 = (\frac{5}{6}, \frac{1}{6})$</td>
<td>$\frac{25}{6}, \frac{5}{6}$</td>
<td>$\frac{1}{6}, \frac{5}{6}$</td>
<td>$\frac{5}{6}, \frac{5}{6}$</td>
</tr>
</tbody>
</table>

- Both players are indifferent among their pure and mixed strategies at equilibrium.
- Players who randomize in a Nash equilibrium are indifferent among their strategies played with positive probability.
- This is a general feature of mixed strategy Nash equilibrium and turns out to be useful.
**Indifference property**

- In games with finite set of strategies, all strategies used in a mixed strategy must yield the same expected utility \( \Rightarrow \) **indifference**

- A mixed strategy can never yield a higher payoff than the best pure strategy

- We can use the indifference property to easily compute mixed strategy Nash equilibria, if we know that such an equilibrium exists (if not, draw best response functions)

- Suppose, Homer chooses TV with the probability \( q \) (and theater with probability \( 1 - q \))

- Marge is then indifferent between TV and theater if:
  \[
  5(1 - q) + 0q = 0(1 - q) + 1q \Leftrightarrow q = \frac{5}{6}
  \]

- Equivalently, Nash equilibrium involves \( p = \frac{1}{6} \)
Example

- Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>l</th>
<th>m</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>3,2</td>
<td>2,1</td>
<td>1,3</td>
</tr>
<tr>
<td>M</td>
<td>2,1</td>
<td>1,5</td>
<td>0,3</td>
</tr>
<tr>
<td>D</td>
<td>1,3</td>
<td>4,2</td>
<td>2,2</td>
</tr>
</tbody>
</table>

- Find best response functions and draw them in a unit square
- Find all Nash equilibria!
Existence of equilibrium: Matching pennies

- Consider again the Matching pennies game!

- Remember that we could not find any Nash equilibria in pure strategies!

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>T</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
</tbody>
</table>

- Show that there is a Nash equilibrium in mixed strategy at $\sigma^* = (\frac{1}{2}; \frac{1}{2})$!

- Is this coincidence or a general feature of Nash equilibria?
Existence of equilibrium in finite games

- Consider the following payoffs for player 1:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
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<tbody>
<tr>
<td>(1-q)</td>
<td></td>
<td>q</td>
</tr>
<tr>
<td>(1-p)</td>
<td>x, -</td>
<td>y, -</td>
</tr>
<tr>
<td>p</td>
<td>z, -</td>
<td>w, -</td>
</tr>
</tbody>
</table>

- There are two important comparisons: x vs. z and y vs. w

- We can define four cases:
  1. \( x > z \) and \( y > w \)
  2. \( x < z \) and \( y < w \)
  3. \( x < z \) and \( y > w \)
  4. \( x > z \) and \( y < w \)

- For simplicity we ignore the remaining cases involving indifference \((x = z \text{ and } y = w)\)
Existence of equilibrium in finite games

- **Case 1:** $U$ is a dominant strategy $\rightarrow p = 0$ for every $q$

- **Case 2:** $D$ is a dominant strategy $\rightarrow p = 1$ for every $q$
Existence of equilibrium in finite games

- **Indifference property:**

\[ x(1 - q) + yq = z(1 - q) + wq \]

\[ q' = \frac{x - z}{x - z + w - y} \]

- **Best-responses in case 3:**

\[ B_1(q) = \begin{cases} 
\{0\} & \text{if } q \in (q', 1] \\
[0, 1] & \text{if } q = q' \\
\{1\} & \text{if } q \in [0, q') 
\end{cases} \]

- **Best-responses in case 4:**

\[ B_1(q) = \begin{cases} 
\{0\} & \text{if } q \in [0, q') \\
[0, 1] & \text{if } q = q' \\
\{1\} & \text{if } q \in (q', 1] 
\end{cases} \]
Existence of equilibrium in finite games

- **Case 3:** $U$ is optimal if $q > q'$ and $D$ is optimal for $q < q'$

- **Case 4:** $U$ is optimal if $q < q'$ and $D$ is optimal for $q > q'$
Existence of equilibrium in finite games

- Arbitrary payoffs for player 2 and analogous computations yield the same four best-response correspondences with rotated axis:
Existence of equilibrium in finite games

- Now consider the 4 best-response correspondences for both players ⇒ in how many cases do best-response correspondence intersect?

- Check all 16 possible pairs of best-response correspondences ⇒ any pair of best-response correspondences has at least one intersection

- The game has at least one Nash equilibrium:
  - a single pure-strategy Nash equilibrium (e.g. Prisoners’ dilemma)
  - a single mixed-strategy Nash equilibrium (e.g. Matching pennies)
  - two pure-strategy Nash equilibria and a single mixed-strategy Nash equilibrium (e.g. Battle of the sexes)

- Mathematical proof for general n-player games with arbitrary finite strategy spaces is beyond the scope of this course!
Nash’s Theorem (1950)

Theorem

In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, if $n$ is finite and $S_i$ is finite for every $i$, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

- This does not help us in finding an equilibrium, but we know that there should be one.

- Nash’s theorem applies to a broad class of games, but not in the presence of infinite strategy spaces (e.g. Cournot competition, relationship game)

- Nash’s Theorem offers sufficient but not necessary conditions for an equilibrium to exist → there are many games not satisfying Nash’s theorem but nonetheless have one or more Nash equilibria
Interpretation of mixed strategy equilibrium

Steady state

- Mixed strategy could be a stochastic steady state
- Players have information about frequencies with which actions were taken in the past
- Each player uses these frequencies to form a belief about other players’ future behavior
- In equilibrium, these frequencies remain constant over time are stable
- It is stochastic, so that for a single play of a game, its prediction is less precise
Interpretation of mixed strategy equilibrium (cont.)

Deliberate decision

- Mixed strategy could be a deliberate decision by a player to introduce randomness into behavior

- e.g. bluffing in poker, governments randomly audit taxpayers, stores randomly offer discounts...

- Critique: randomization is a ”bizarre” description of deliberate strategies

- Problem also that in a mixed strategy equilibrium each player is indifferent between mixed strategies yielding the same expected payoff
Interpretation of mixed strategy equilibrium (cont.)

Perturbed game

- A game is seen as a frequently occurring situation in which players’ preferences are subject to small random variations.
- In each occurrence, each player knows her own preferences but not those of the other players.
- A mixed strategy equilibrium is a summary of the frequencies with which the players choose their actions over time.

Beliefs

- A mixed strategy equilibrium is a profile of beliefs $\beta$, in which every $\beta_i$ is the common belief of all the other players about player $i$’s actions.
- For each player $i$, each $\beta_i$ is optimal given $\beta_{-i}$.
- In this interpretation, an equilibrium is a steady state of beliefs, not actions.
Application:Credence good

- There are two types of problems, major and minor - denote the fraction of major problems by $r \in [0, 1]$
- An expert knows the type of the problem, the consumer only knows the probability $r$
- An expert may either recommend a major or a minor repair (regardless of the type of the problem)
- A consumer may either accept expert’s recommendation or reject and seek other remedies - consumer always accept a minor repair
- Major repairs solve major and minor problems, and experts always recommend a major repair for a major problem
- An expert obtains profit $\pi > 0$ from selling a minor repair to minor problems and major repairs to major problems. The expert obtains a profit $\pi' > \pi$ by selling a major repair to a minor problem
- A consumer pays $E$ for a major repair and $I < E$ for a minor repair. If the consumer seeks an alternative remedy, she pays $E' > E$ for a major problem and $I' > I$ if it is minor. We also assume that $E > I'$
Write down this game in normal-form.

Find best-response functions for both players.

Compute all pure and mixed-strategy Nash equilibria.

How do equilibria change if...

- major problems occur less often
- major repairs become cheaper relative to minor repairs
- the profit $\pi'$ from fixing a minor problem with a major repair is falling